FACTORIZATIONS OF CONTRACTIONS

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Dedicated to Professor Rajendra Bhatia on the occasion of his 65th birthday

ABSTRACT. The celebrated Sz.-Nagy and Foias theorem asserts that every pure contraction is unitarily equivalent to an operator of the form $P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ where $\mathcal{Q}$ is a M_z^*-invariant subspace of a $\mathcal{D}$-valued Hardy space $H^2_{\mathcal{D}}(\mathbb{D})$, for some Hilbert space $\mathcal{D}$.

On the other hand, the celebrated theorem of Berger, Coburn and Lebow on pairs of commuting isometries can be formulated as follows: a pure isometry V on a Hilbert space $\mathcal{H}$ is a product of two commuting isometries V_1 and V_2 in $\mathcal{B}(\mathcal{H})$ if and only if there exist a Hilbert space $\mathcal{E}$, a unitary U in $\mathcal{B}(\mathcal{E})$ and an orthogonal projection P in $\mathcal{B}(\mathcal{E})$ such that (V, V_1, V_2) and (M_z, M_Φ, M_Ψ) on $H^2_{\mathcal{E}}(\mathbb{D})$ are unitarily equivalent, where

$$\Phi(z) = (P + zP^{\perp})U^* \quad \text{and} \quad \Psi(z) = U(P^{\perp} + zP) \qquad (z \in \mathbb{D}).$$

In this context, it is natural to ask whether similar factorization results hold true for pure contractions. The purpose of this paper is to answer this question. More particularly, let T be a pure contraction on a Hilbert space $\mathcal{H}$ and let $P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ be the Sz.-Nagy and Foias representation of T for some canonical $\mathcal{Q} \subseteq H^2_{\mathcal{D}}(\mathbb{D})$. Then $T = T_1 T_2$, for some commuting contractions T_1 and T_2 on $\mathcal{H}$, if and only if there exist $\mathcal{B}(\mathcal{D})$-valued polynomials φ and ψ of degree ≤ 1 such that $\mathcal{Q}$ is a joint (M_φ^*, M_ψ^*)-invariant subspace,

$$P_{\mathcal{Q}} M_z|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\psi\varphi}|_{\mathcal{Q}} \text{ and } (T_1, T_2) \cong (P_{\mathcal{Q}} M_\varphi|_{\mathcal{Q}}, P_{\mathcal{Q}} M_\psi|_{\mathcal{Q}}).$$

Moreover, there exist a Hilbert space $\mathcal{E}$ and an isometry $V \in \mathcal{B}(\mathcal{D}; \mathcal{E})$ such that

$$\varphi(z) = V^* \Phi(z) V \text{ and } \psi(z) = V^* \Psi(z) V \qquad (z \in \mathbb{D}),$$

where the pair (Φ, Ψ), as defined above, is the Berger, Coburn and Lebow representation of a pure pair of commuting isometries on $H^2_{\mathcal{E}}(\mathbb{D})$. As an application, we obtain a sharper von Neumann inequality for commuting pairs of contractions.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and V be an isometry on $\mathcal{H}$. It is a classical result, due to von Neumann and Wold (cf. [13]), that V is unitarily equivalent to $M_z \oplus U$ where M_z is the shift operator on an $\mathcal{E}$-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$, for some Hilbert space $\mathcal{E}$, and U is a unitary operator on $\mathcal{H}_u$, where

$$\mathcal{H}_u = \bigcap_{m=0}^{\infty} V^m \mathcal{H}.$$

We say that V is *pure* if $\mathcal{H}_u = \{0\}$, or, equivalently, if $V^{*m} \to 0$ in the strong operator topology (that is, $\|V^{*m} h\| \to 0$ as $m \to \infty$ for all $h \in \mathcal{H}$). Pure isometry, that is, shift operators on

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vector-valued Hardy spaces play an important role in the study of general operators that stems from the following result (see [13, 14]):

Theorem 1.1. *(Sz.-Nagy and Foias) Let T be a pure contraction on a Hilbert space $\mathcal{H}$. Then T and $P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$ are unitarily equivalent, where $\mathcal{Q}$ is a closed M_z^*-invariant subspace of a vector-valued Hardy space $H^2_{\mathcal{D}}(\mathbb{D})$.*

Here the $\mathcal{D}$-valued Hardy space over $\mathbb{D}$, denoted by $H^2_{\mathcal{D}}(\mathbb{D})$, is defined by

$$H^2_{\mathcal{D}}(\mathbb{D}) := \{f = \sum_{k\in\mathbb{N}} \eta_k z^k \in \mathcal{O}(\mathbb{D}, \mathcal{D}) : \eta_j \in \mathcal{D}, j \in \mathbb{N}, \|f\|^2 := \sum_{k\in\mathbb{N}} \|\eta_k\|^2 < \infty\}.$$

Recall that a contraction T on a Hilbert space $\mathcal{H}$ is *pure* (cf. [16]) if $T^{*m} \to 0$ as $m \to \infty$ in the strong operator topology. Also note that, in the above theorem, one can choose the coefficient Hilbert space $\mathcal{D}$ as $\overline{\text{ran}}(I - TT^*)$ (see [13]).

In contrast with the von-Neumann and Wold decomposition theorem for isometries, the structure of commuting n-tuples of isometries, $n \geq 2$, is much more complicated and very little, in general, is known (see [3, 4, 5, 6, 7, 11, 12, 17, 18, 15]). However, for pure pairs of commuting isometries, the problem is more tractable.

A pair of commuting isometries (V_1, V_2) on a Hilbert space $\mathcal{H}$ is said to be *pure* if V_1V_2 is a pure isometry, that is,

$$\bigcap_{m=0}^{\infty} V_1^m V_2^m \mathcal{H} = \{0\}.$$

With this as motivation, a pair of commuting contractions (T_1, T_2) is said to be *pure* if T_1T_2 is a pure contraction.

The concept of pure pair of commuting isometries introduced by Berger, Coburn and Lebow [6] is an important development in the study of representation and Fredholm theory for C^*-algebras generated by commuting isometries. They showed that a pair of commuting isometries (V_1, V_2) on a Hilbert space $\mathcal{H}$ is pure if and only if there exist a Hilbert space $\mathcal{E}$, a unitary U in $\mathcal{B}(\mathcal{E})$ and an orthogonal projection in $\mathcal{B}(\mathcal{E})$ such that (V_1, V_2) on $\mathcal{H}$ and (M_Φ, M_Ψ) on $H^2_{\mathcal{E}}(\mathbb{D})$ are jointly unitarily equivalent, where

$$\Phi(z) = (P + zP^\perp)U^* \quad \text{and} \quad \Psi(z) = U(P^\perp + zP) \qquad (z \in \mathbb{D}). \tag{1.1}$$

Moreover, it follows that

$$M_\Phi M_\Psi = M_\Psi M_\Phi = M_z,$$

and V_1V_2 on $\mathcal{H}$ and M_z on $H^2_{\mathcal{E}}(\mathbb{D})$ are unitarily equivalent (see also [5, 10]). More precisely, if $\Pi : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D})$ denotes the unitary map, implemented by the Wold and von Neumann decomposition of the pure isometry V_1V_2 with $\mathcal{E} = \text{ran}(I - V_1V_2V_1^*V_2^*)$ (cf. [16]), then

$$\Pi V_1 = M_\Phi \Pi, \quad \text{and} \quad \Pi V_2 = M_\Psi \Pi.$$

In what follows, for a triple $(\mathcal{E}, U, P)$ as above we let $\Phi, \Psi \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ denote the isometric multipliers as defined in (1.1). We call (M_Φ, M_Ψ) the pair of isometries associated with the triple $(\mathcal{E}, U, P)$.

Our work is motivated by the following equivalent interpretations of the Berger, Coburn and Lebow's characterizations of pure pairs of commuting isometries:

(I) Let (V_1, V_2) be a pure pair of commuting isometries and let M_z on $H^2_{\mathcal{E}}(\mathbb{D})$ be the von Neumann and Wold decomposition representation of V_1V_2. Then there exist a unitary U and an orthogonal projection P in $\mathcal{B}(\mathcal{E})$ such that the representations of V_1 and V_2 in $\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))$ are given by M_Φ and M_Ψ, respectively.

(II) Let (X, Y) be a pair of commuting isometries in $\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))$. Moreover, let X and Y are Toeplitz operators [13] with analytic symbols from $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$. Then $M_z = XY$ if and only if there exist a unitary U and an orthogonal projection P in $\mathcal{B}(\mathcal{E})$ such that $(X, Y) = (M_\Phi, M_\Psi)$.

In this paper we shall obtain similar results for pure pairs of commuting contractions acting on Hilbert spaces. More specifically, summarizing Theorems 3.1, 3.2 and 4.1 and Corollary 4.2, we have the following: Let T be a pure contraction and (T_1, T_2) be a pair of commuting contractions on a Hilbert space $\mathcal{H}$. Let $\mathcal{Q}$ be the Sz.-Nagy and Foias representation of T, that is, $\mathcal{Q}$ is a M_z^*-invariant subspace of a vector-valued Hardy space $H^2_{\mathcal{D}}(\mathbb{D})$ and T and $P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$ are unitarily equivalent (see Theorem 1.1 and Section 2) where $\mathcal{D} = \overline{\mathrm{ran}}(I - TT^*)$. Then the following are equivalent:

(i) $T = T_1T_2$.

(ii) There exist a triple $(\mathcal{E}, U, P)$ and a joint $(M_z^*, M_\Phi^*, M_\Psi^*)$-invariant subspace $\tilde{\mathcal{Q}}$ of $H^2_{\mathcal{E}}(\mathbb{D})$ such that

$$T_1 \cong P_{\tilde{\mathcal{Q}}}M_\Phi|_{\tilde{\mathcal{Q}}}, T_2 \cong P_{\tilde{\mathcal{Q}}}M_\Psi|_{\tilde{\mathcal{Q}}}, T \cong P_{\tilde{\mathcal{Q}}}M_z|_{\tilde{\mathcal{Q}}} \text{ and } M_\Phi M_\Psi = M_\Psi M_\Phi = M_z.$$

In other words, (T_1, T_2, T) on $\mathcal{H}$ dilates to (M_Φ, M_Ψ, M_z) on $H^2_{\mathcal{E}}(\mathbb{D})$.

(iii) There exist $\mathcal{B}(\mathcal{D})$-valued polynomials φ and ψ of degree ≤ 1 such that $\mathcal{Q}$ is a joint (M_φ^*, M_ψ^*)-invariant subspace,

$$P_{\mathcal{Q}}M_z|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\psi\varphi}|_{\mathcal{Q}},$$

and

$$(T_1, T_2) \cong (P_{\mathcal{Q}}M_\varphi|_{\mathcal{Q}}, P_{\mathcal{Q}}M_\psi|_{\mathcal{Q}}).$$

In particular, if $T = T_1T_2$ is pure then the Sz.-Nagy and Foias representations of T_1 and T_2 on $\mathcal{Q}$ are given by $P_{\mathcal{Q}}M_\varphi|_{\mathcal{Q}}$ and $P_{\mathcal{Q}}M_\psi|_{\mathcal{Q}}$, respectively. Moreover, it turns out that the pair (M_φ, M_ψ) can be chosen as

$$\varphi(z) = V^*\Phi(z)V \text{ and } \psi(z) = V^*\Psi(z)V \qquad (z \in \mathbb{D}),$$

where $V \in \mathcal{B}(\mathcal{D}; \mathcal{E})$ is an isometry and Φ, Ψ are the isometric multipliers associated with the triple $(\mathcal{E}, U, P)$ from condition (ii). As an application of our results we give a sharper von Neumann inequality for commuting pairs of contractions: Let (T_1, T_2) be a commuting pair of contractions on $\mathcal{H}$. Also assume that T_1T_2 is a pure contraction and rank $(I_{\mathcal{H}} - T_iT_i^*) < \infty$, $i = 1, 2$. Then there exists a variety V in $\mathbb{D}^2$ such that

$$\|p(T_1, T_2)\| \leq \sup_{(z_1,z_2)\in V} |p(z_1, z_2)| \qquad (p \in \mathbb{C}[z_1, z_2]).$$

The plan of the paper is the following. Section 2 contains some preliminaries and a key dilation result. In Section 3, we prove that a pure pair of commuting contractions always dilates to a pure pair of commuting isometries. Our construction is more explicit for pairs of contractions with finite dimensional defect spaces. In Section 4, we obtain explicit representations of commuting and contractive factors of a pure contraction in its corresponding

Sz.-Nagy and Foias space. In the last section, we consider von Neumann inequality for pure pair of commuting contractions.

2. Preliminaries

In this section, we set notation and definitions and discuss some preliminaries. Also we prove a basic dilation result in Theorem 2.1. This result will play a fundamental role throughout the remainder of the paper.

Let T be a contraction on a Hilbert space $\mathcal{H}$ (that is, $\|Tf\| \leq \|f\|$ for all $f \in \mathcal{H}$ or, equivalently, if $I_{\mathcal{H}} - TT^* \geq 0$) and let $\mathcal{E}$ be a Hilbert space. Then M_z on $H^2_{\mathcal{E}}(\mathbb{D})$ is called an isometric dilation of T (cf. [16]) if there exists an isometry $\Gamma : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D})$ such that

$$\Gamma T^* = M_z^* \Gamma.$$

Similarly, a pair of commuting operators (U_1, U_2) on $\mathcal{K}$ is said to be a dilation of a commuting pair of operators (T_1, T_2) on $\mathcal{H}$ if there exists an isometry $\Gamma : \mathcal{H} \to \mathcal{K}$ such that

$$\Gamma T_j^* = U_j^* \Gamma \qquad (j = 1, 2).$$

Note that, in this case, $\mathcal{Q} := \mathrm{ran}\Gamma$ is a joint (U_1^*, U_2^*)-invariant subspace of $\mathcal{K}$ and

$$T_j \cong P_{\mathcal{Q}} U_j|_{\mathcal{Q}} \qquad (j = 1, 2).$$

Now let T be a contraction on a Hilbert space $\mathcal{H}$. Set

$$\mathcal{D}_T = \overline{\mathrm{ran}}(I_{\mathcal{H}} - TT^*), \qquad D_T = (I_{\mathcal{H}} - TT^*)^{\frac{1}{2}}.$$

If in addition, T is pure then M_z on $H^2_{\mathcal{D}_T}(\mathbb{D})$, induced by the isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_T}(\mathbb{D})$, is an isometric dilation of T (cf. [16]), where

$$(\Pi h)(z) = D_T(I_{\mathcal{H}} - zT^*)^{-1} h \qquad (z \in \mathbb{D}, h \in \mathcal{H}). \tag{2.1}$$

In particular, this yields a proof of Theorem 1.1 that every pure contraction is unitarily equivalent to the compression of M_z to an M_z^*-invariant closed subspace of a vector-valued Hardy space.

It is also important to note that the above dilation is minimal, that is,

$$H^2_{\mathcal{D}_T}(\mathbb{D}) = \overline{\mathrm{span}}\{z^m \Pi f : m \in \mathbb{N}, f \in \mathcal{H}\}, \tag{2.2}$$

and hence unique in an appropriate sense (see [13]).

Our considerations will also rely on the techniques of transfer functions (cf. [8]). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2),$$

be a unitary operator. Then the $\mathcal{B}(\mathcal{H}_1)$-valued analytic function τ_U on $\mathbb{D}$ defined by

$$\tau_U(z) := A + zB(I - zD)^{-1}C \qquad (z \in \mathbb{D}),$$

is called the *transfer function* of U. Using $U^*U = I$, a standard and well known computation yields (cf. [8])

$$I - \tau_U(z)^* \tau_U(z) = (1 - |z|^2) C^* (I - \bar{z} D^*)^{-1} (I - zD)^{-1} C \qquad (z \in \mathbb{D}). \tag{2.3}$$

However, in this paper, we will mostly deal with transfer functions corresponding to unitary matrices of the form $U = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. In this case, it follows (see (2.3)) from the identity

$$I - \tau_U(z)^*\tau_U(z) = (1-|z|^2)C^*C \qquad (z \in \mathbb{D})$$

that τ_U is a $B(\mathcal{H}_1)$-valued inner function [13].

Now let (T_1, T_2) be a pair of commuting contractions. Since

$$(I_{\mathcal{H}} - T_1T_1^*) + T_1(I_{\mathcal{H}} - T_2T_2^*)T_1^* = T_2(I_{\mathcal{H}} - T_1T_1^*)T_2^* + (I_{\mathcal{H}} - T_2T_2^*),$$

it follows that

$$\|D_{T_1}h\|^2 + \|D_{T_2}T_1^*h\|^2 = \|D_{T_1}T_2^*h\|^2 + \|D_{T_2}h\|^2 \quad (h \in \mathcal{H}).$$

Thus

$$U : \{D_{T_1}h \oplus D_{T_2}T_1^*h : h \in \mathcal{H}\} \to \{D_{T_1}T_2^*h \oplus D_{T_2}h : h \in \mathcal{H}\}$$

defined by

$$U\left(D_{T_1}h, D_{T_2}T_1^*h\right) = (D_{T_1}T_2^*h, D_{T_2}h) \qquad (h \in \mathcal{H}), \tag{2.4}$$

is an isometry. This operator will play a very important role in the sequel.

We now formulate the main theorem of this section, a result which will play a very important part in our considerations later on. Here the proof is similar in spirit to the main dilation result of [8].

Let $\mathcal{H}$ and $\mathcal{E}$ be Hilbert spaces and let (S, T) be a pair of commuting contractions on $\mathcal{H}$. Let T be pure and $V \in \mathcal{B}(\mathcal{D}_T; \mathcal{E})$ be an isometry. Then the isometric dilation of T, $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_T}(\mathbb{D})$ as defined in (2.1), allows us to define an isometry $\Pi_V \in \mathcal{B}(\mathcal{H}; H^2_{\mathcal{E}}(\mathbb{D}))$ by setting

$$\Pi_V := (I_{H^2(\mathbb{D})} \otimes V)\Pi.$$

It is easy to check that

$$\Pi_V T^* = (M_z^* \otimes I_{\mathcal{E}})\Pi_V,$$

and hence we conclude that M_z on $H^2_{\mathcal{E}}(\mathbb{D})$ is an isometric dilation of T. In particular, $\mathcal{Q} = \Pi_V\mathcal{H}$ is a M_z^*-invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ and $T \cong P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$.

Theorem 2.1. *With the notations as above, let*

$$U = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} : \mathcal{E} \oplus \mathcal{D}_S \to \mathcal{E} \oplus \mathcal{D}_S,$$

be a unitary operator such that

$$U(VD_Th, D_ST^*h) = (VD_TS^*h, D_Sh) \qquad (h \in \mathcal{H}).$$

We denote by $\Phi(z) = A^* + zC^*B^*$ the transfer function of U^*. Then Φ is a $\mathcal{B}(\mathcal{E})$-valued inner function and

$$\Pi_V S^* = M_\Phi^* \Pi_V.$$

In particular, $\mathcal{Q} = \Pi_V\mathcal{H}$ is a joint (M_z^*, M_Φ^*)-invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ and

$$T^* \cong M_z^*|_{\mathcal{Q}} \quad \text{and } S^* \cong M_\Phi^*|_{\mathcal{Q}}.$$

Proof. We only need to prove that $\Pi_V S^* = M_\Phi^* \Pi_V$. Now for each $h \in \mathcal{H}$ we have the equality

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} VD_T h \\ D_S T^* h \end{bmatrix} = \begin{bmatrix} VD_T S^* h \\ D_S h \end{bmatrix},$$

that is,

$$VD_T S^* h = AVD_T h + BD_S T^* h, \quad \text{and} \quad D_S h = CVD_T h.$$

This implies

$$VD_T S^* = AVD_T + BCVD_T T^*.$$

Now if $n \geq 1$, $h \in \mathcal{H}$ and $\eta \in \mathcal{E}$, then

$$\begin{aligned} \langle M_\Phi^* \Pi_V h, z^n \eta \rangle &= \langle (I \otimes V) D_T (I - zT^*)^{-1} h, (A^* + zC^* B^*)(z^n \eta) \rangle \\ &= \langle VD_T T^{*n} h, A^* \eta \rangle + \langle VD_T T^{*(n+1)} h, C^* B^* \eta \rangle \\ &= \langle (AVD_T + BCVD_T T^*)(T^{*n} h), \eta \rangle \\ &= \langle VD_T S^* (T^{*n} h), \eta \rangle. \end{aligned}$$

On the other hand, since

$$\langle \Pi_V S^* h, z^n \eta \rangle = \langle VD_T (I - zT^*)^{-1} S^* h, z^n \eta \rangle = \langle (VD_T S^*)(T^{*n} h), \eta \rangle,$$

we get $\Pi_V S^* = M_\Phi^* \Pi_V$. This completes the proof. □

3. Dilating to pure isometries

In this section we prove that a pure pair of commuting contractions dilates to a pure pair of commuting isometries. We describe the construction of dilations more explicitly in the case of finite dimensional defect spaces.

Theorem 3.1. *Let (T_1, T_2) be a pure pair of commuting contractions on $\mathcal{H}$ and dim $\mathcal{D}_{T_j} < \infty$, $j = 1, 2$. Then (T_1, T_2) dilates to a pure pair of commuting isometries.*

Proof. Set $\mathcal{E} := \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$ and $T := T_1 T_2$. Let $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_T}(\mathbb{D})$ be the isometric dilation of T as defined in (2.1). Now observe that the equality

$$I - TT^* = I - T_1 T_2 T_1^* T_2^* = (I - T_1 T_1^*) + T_1 (I - T_2 T_2^*) T_1^*,$$

implies that the operator $V \in \mathcal{B}(\mathcal{D}_T; \mathcal{E})$ defined by

$$V(D_T h) = (D_{T_1} h, D_{T_2} T_1^* h) \qquad (h \in \mathcal{H}),$$

is an isometry. Consequently,

$$\Pi_V := (I_{H^2(\mathbb{D})} \otimes V)\Pi : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D}) \tag{3.1}$$

is an isometric dilation of T, and hence $T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ where $\mathcal{Q} = \Pi_V \mathcal{H}$ is a M_z^*-invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ (see the proof of Theorem 2.1). Let $\iota_j : \mathcal{D}_{T_j} \to \mathcal{E}$, $j = 1, 2$, be the inclusion maps, defined by

$$\iota_1(h_1) = (h_1, 0) \quad \text{and} \quad \iota_2(h_2) = (0, h_2) \qquad (h_1 \in \mathcal{D}_{T_1}, h_2 \in \mathcal{D}_{T_2}).$$

Then $P := \iota_2\iota_2^* \in \mathcal{B}(\mathcal{E})$ is the orthogonal projection onto $\mathcal{D}_{T_2}$, that is,

$$P(h_1, h_2) = (0, h_2) \qquad ((h_1, h_2) \in \mathcal{E}).$$

Thus, $\iota_1\iota_1^* = P^\perp$ is the orthogonal projection onto $\mathcal{D}_{T_1}$, and so

$$\begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} : \mathcal{E} \oplus \mathcal{D}_{T_1} \to \mathcal{E} \oplus \mathcal{D}_{T_1}$$

is a unitary. Now since $\dim \mathcal{E} < \infty$, it follows that the isometry U, as defined in (2.4), extends to a unitary, denoted again by U, on $\mathcal{E}$. In particular, there exists a unitary operator U on $\mathcal{E}$ such that

$$U(D_{T_1}T_2^*h, D_{T_2}h) = (D_{T_1}h, D_{T_2}T_1^*h) \qquad (h \in \mathcal{H}).$$

Then

$$U_1 = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} = \begin{bmatrix} UP & U\iota_1 \\ \iota_1^* & 0 \end{bmatrix},$$

is a unitary operator in $\mathcal{B}(\mathcal{E} \oplus \mathcal{D}_{T_1})$. Moreover, for all $h \in \mathcal{H}$, we have

$$\begin{aligned} U_1\big(V(D_T h), D_{T_1}T^*h\big) &= U_1\big(D_{T_1}h, D_{T_2}T_1^*h, D_{T_1}T_1^*T_2^*h\big) \\ &= \big(U(D_{T_1}T_1^*T_2^*h, D_{T_2}T_1^*h), D_{T_1}h\big) \\ &= \big(D_{T_1}T_1^*h, D_{T_2}(T_1^*)^2h, D_{T_1}h\big) \\ &= \big(V(D_T T_1^*h), D_{T_1}h\big). \end{aligned}$$

Consequently, by Theorem 2.1 we have

$$\Pi_V T_1^* = M_\Phi^* \Pi_V,$$

where

$$\Phi(z) = PU^* + z\iota_1\iota_1^*U^* = (P + zP^\perp)U^* \qquad (z \in \mathbb{D}),$$

is the transfer function of the unitary operator U_1^*. Similarly, if we define a unitary $U_2 \in \mathcal{B}(\mathcal{E} \oplus \mathcal{D}_{T_2})$ by

$$U_2 = \begin{bmatrix} P^\perp & \iota_2 \\ \iota_2^* & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P^\perp U^* & \iota_2 \\ \iota_2^*U^* & 0 \end{bmatrix},$$

then

$$U_2\big(V(D_T h), D_{T_2}T^*h\big) = \big(V(D_T T_2^*h), D_{T_2}h\big) \qquad (h \in \mathcal{H}),$$

and hence by Theorem 2.1, we have

$$\Pi_V T_2^* = M_\Psi^* \Pi_V,$$

where

$$\Psi(z) = UP^\perp + zU\iota_2\iota_2^* = U(P^\perp + zP),$$

is the transfer function for the unitary operator U_2^*. This completes the proof that the pure pair of commuting isometries (M_Φ, M_Ψ) on $H^2_{\mathcal{E}}(\mathbb{D})$ corresponding to the triple $(\mathcal{E}, U, P)$ dilates (T_1, T_2). $\square$

We will now go on to give a proof of the general result. The proof is essentially the same as the previous theorem except the constructions of unitary operators and inclusion maps.

Theorem 3.2. *Let (T_1, T_2) be a pure pair of commuting contractions on $\mathcal{H}$. Then (T_1, T_2) dilates to a pure pair of commuting isometries.*

Proof. Let $\dim \mathcal{D}_{T_1} = \infty$, or $\dim \mathcal{D}_{T_2} = \infty$ and $\mathcal{D}$ be an infinite dimensional Hilbert space. Set $\mathcal{E} := (\mathcal{D} \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2}$. We now define inclusion maps $\iota_1 : \mathcal{D} \oplus \mathcal{D}_{T_1} \to \mathcal{E}$ and $\iota_2 : \mathcal{D}_{T_2} \to \mathcal{E}$ by

$$\iota_1(h, h_1) = (h, h_1, 0) \quad \text{and} \quad \iota_2 h_2 = (0, 0, h_2), \quad (h \in \mathcal{D}, h_1 \in \mathcal{D}_{T_1}, h_2 \in \mathcal{D}_{T_2})$$

respectively, and an isometric embedding $V \in \mathcal{B}(\mathcal{D}_T; \mathcal{E})$ by

$$V D_T h = (0, D_{T_1} h, D_{T_2} T_1^* h) \quad (h \in \mathcal{H}).$$

We also define the orthogonal projection P by $P = \iota_2 \iota_2^*$. Therefore

$$P(h_1, h_2, h_3) = (0, 0, h_3) \qquad ((h_1, h_2, h_3) \in \mathcal{E}).$$

Finally, since

$$U_{\mathcal{D}} \left(0_{\mathcal{D}}, D_{T_1} h, D_{T_2} T_1^* h\right) = (0_{\mathcal{D}}, D_{T_1} T_2^* h, D_{T_2} h) \qquad (h \in \mathcal{H}),$$

defines an isometry from $\{0_{\mathcal{D}}\} \oplus \{D_{T_1} h \oplus D_{T_2} T_1^* h : h \in \mathcal{H}\}$ to $\{0_{\mathcal{D}}\} \oplus \{D_{T_1} T_2^* h \oplus D_{T_2} h : h \in \mathcal{H}\}$, we can therefore extend $U_{\mathcal{D}}$ to a unitary, denoted again by $U_{\mathcal{D}}$, acting on $\mathcal{E}$. With these notations we define unitary operators

$$U_1 = \begin{bmatrix} U_{\mathcal{D}} P & U_{\mathcal{D}} \iota_1 \\ \iota_1^* & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{T_1})) \quad \text{and } U_2 = \begin{bmatrix} P^{\perp} U_{\mathcal{D}}^* & \iota_2 \\ \iota_2^* U_{\mathcal{D}}^* & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{E} \oplus \mathcal{D}_{T_2}).$$

The rest of the proof proceeds in the same way as in Theorem 3.1. This completes the proof. ☐

The main inconvenience of our approach seems to be the nonuniqueness of the triple $(\mathcal{E}, U, P)$. This issue is closely related to the nonuniqueness of Ando dilation [2] and solutions of commutant lifting theorem [9].

It is also important to note that Theorem 3.2 is a sharper version of Ando dilation [2] for pure pairs of commuting contractions. More precisely, one can dilate a pure pair of commuting contractions to a pure pair of commuting isometries, in the sense of Berger, Coburn and Lebow. In the context of concrete isometric dilations for commuting pairs of pure contractions, see [1] and [8].

4. Factorizations

Let (T_1, T_2) be a pair of commuting contractions on $\mathcal{H}$ and $T = T_1 T_2$ be a pure contraction. Then by Theorem 1.1 we can realize T as $P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ where $\mathcal{Q} = \operatorname{ran}\Pi = \Pi\mathcal{H}$ is the Sz.-Nagy and Foias model space and $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_T}(\mathbb{D})$ is the minimal isometric dilation of T (see (2.2)).

In this section we will show that T_1 and T_2 can be realized as compressions of two $\mathcal{B}(\mathcal{D}_T)$-valued polynomials of degree ≤ 1 in the Sz.-Nagy and Foias model space $\mathcal{Q}$ of the pure contraction T.

Let $\Pi_V : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D})$ be the isometric dilation as in Theorems 3.1 and 3.2, that is,

$$\Pi_V T_1^* = M_{\Phi}^* \Pi_V \quad \text{and} \quad \Pi_V T_2^* = M_{\Psi}^* \Pi_V.$$

Then it follows from 3.1 that

$$\Pi T_1^* = (I \otimes V^*) M_\Phi^* (I \otimes V)\Pi = M_\varphi^* \Pi,$$

where

$$\varphi(z) = V^* \Phi(z) V \qquad (z \in \mathbb{D}),$$

and $V \in \mathcal{B}(\mathcal{D}_T; \mathcal{E})$ is an isometry. Similarly, we derive

$$\Pi T_2^* = (I \otimes V^*) M_\Psi^* (I \otimes V)\Pi = M_\psi^* \Pi,$$

where

$$\psi(z) = V^* \Psi(z) V \qquad (z \in \mathbb{D}).$$

In particular, ran Π is a joint (M_φ^*, M_ψ^*)-invariant subspace and by construction of Π it follows that

$$\Pi T^* = M_z^* \Pi,$$

and ran Π is a also a M_z^*-invariant subspace of $H^2_{\mathcal{D}_T}(\mathbb{D})$. We have thus proved the following theorem.

Theorem 4.1. *Let T be a pure contraction on a Hilbert space $\mathcal{H}$ and let $T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ be the Sz.-Nagy and Foias representation of T as in Theorem 1.1. If $T = T_1 T_2$, for some commuting pair of contractions (T_1, T_2) on $\mathcal{H}$, then there exists $\mathcal{B}(\mathcal{D}_T)$-valued polynomials φ and ψ of degree ≤ 1 such that $\mathcal{Q}$ is a joint (M_φ^*, M_ψ^*)-invariant subspace and*

$$(T_1, T_2) \cong (P_{\mathcal{Q}} M_\varphi|_{\mathcal{Q}}, P_{\mathcal{Q}} M_\varphi|_{\mathcal{Q}}).$$

In particular,

$$P_{\mathcal{Q}} M_z|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\varphi\phi}|_{\mathcal{Q}}.$$

It is important to note that $P_{\mathcal{Q}} M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\psi\varphi}|_{\mathcal{Q}}$, even though, in general

$$\varphi\psi \neq \psi\varphi.$$

A reformulation of Theorem 4.1 is the following:

Corollary 4.2. *Let T be a pure contraction on a Hilbert space $\mathcal{H}$ and let $T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ be the Sz.-Nagy and Foias representation of T as in Theorem 1.1. Then $T = T_1 T_2$, for some commuting pair of contractions (T_1, T_2) on $\mathcal{H}$, if and only if there exist $\mathcal{B}(\mathcal{D}_T)$-valued polynomials φ and ψ of degree ≤ 1 such that $\mathcal{Q}$ is a joint (M_φ^*, M_ψ^*)-invariant subspace,*

$$P_{\mathcal{Q}} M_z|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\psi\varphi}|_{\mathcal{Q}},$$

and

$$(T_1, T_2) \cong (P_{\mathcal{Q}} M_\varphi|_{\mathcal{Q}}, P_{\mathcal{Q}} M_\psi|_{\mathcal{Q}}).$$

Moreover, there exists a triple $(\mathcal{E}, U, P)$ and an isometry $V \in \mathcal{B}(\mathcal{D}_T; \mathcal{E})$ such that

$$\varphi(z) = V^* \Phi(z) V \text{ and } \psi(z) = V^* \Psi(z) V \qquad (z \in \mathbb{D}).$$

5. von Neumann inequality

In this section we consider the von Neumann inequality for pure pair of commuting contractions with finite dimensional defect spaces. We show that for such a pair there exists a variety in the bidisc where the von Neumann inequality holds.

Theorem 5.1. *Let (T_1, T_2) be a pure pair of commuting contractions on $\mathcal{H}$ and dim $\mathcal{D}_{T_i} < \infty$, $i = 1, 2$. Then there exists an algebraic variety V in $\overline{\mathbb{D}}^2$ such that*

$$\|p(T_1, T_2)\| \leq \sup_{(z_1,z_2)\in V} |p(z_1, z_2)| \qquad (p \in \mathbb{C}[z_1, z_2]).$$

Moreover, if $m = \dim(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2})$, then there exists a pure pair of commuting isometries (M_Φ, M_Ψ) on $H^2_{\mathbb{C}^m}(\mathbb{D})$ such that

$$V = \{(z_1, z_2) \in \overline{\mathbb{D}}^2 : \det(\Phi(z_1 z_2) - z_1 I) = 0 \text{ and } \det(\Psi(z_1 z_2) - z_2 I) = 0\}.$$

Proof. Let $\mathcal{E} = \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$. Then by Theorem 3.1, there exists a pure pair of commuting isometries (M_Φ, M_Ψ) on $H^2_{\mathcal{E}}(\mathbb{D})$ and a joint (M^*_Φ, M^*_Ψ)-invariant subspace $\mathcal{Q}$ of $H^2_{\mathcal{E}}(\mathbb{D})$ such that $T_1 \cong P_{\mathcal{Q}} M_\Phi|_{\mathcal{Q}}$ and $T_2 \cong P_{\mathcal{Q}} M_\Psi|_{\mathcal{Q}}$. Then for each $p \in \mathbb{C}[z_1, z_2]$, we have

$$\begin{aligned}
\|p(T_1, T_2)\|_{\mathcal{B}(\mathcal{H})} &= \|P_{\mathcal{Q}} p(M_\Phi, M_\Psi)|_{\mathcal{Q}}\|_{\mathcal{B}(\mathcal{Q})} \qquad (5.1)\\
&\leq \|p(M_\Phi, M_\Psi)\|_{\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))}\\
&= \|M_{p(\Phi,\Psi)}\|_{\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))}\\
&\leq \sup_{z\in\mathbb{T}} \|p\left(\Phi(z), \Psi(z)\right)\|_{\mathcal{B}(\mathcal{E})}\\
&\leq \sup\{|p(\lambda_1, \lambda_2)| : (\lambda_1, \lambda_2) \in \sigma(\Phi(z), \Psi(z)), z \in \mathbb{T}\}, \qquad (5.2)
\end{aligned}$$

where we denote $\sigma(\Phi(z), \Psi(z))$ by the joint spectrum of the commuting pair of unitary matrices $(\Phi(z), \Psi(z))$, $z \in \mathbb{T}$. Now observe that if $(\lambda_1, \lambda_2) \in \sigma(\Phi(z), \Psi(z))$ for some $z \in \mathbb{T}$, then there exists a non-zero $h \in \mathcal{E}$ such that $\Phi(z)h = \lambda_1 h$ and $\Psi(z)h = \lambda_2 h$. Then $zh = \Phi(z)\Psi(z)h = \lambda_1\lambda_2 h$ and hence $z = \lambda_1\lambda_2$. With this observation we have

$$\{(\lambda_1, \lambda_2) \in \sigma(\Phi(z), \Psi(z)) : z \in \mathbb{T}\} \subset \partial V,$$

where

$$V_1 = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}}^2 : \det(\Phi(\lambda_1\lambda_2) - \lambda_1 I) = 0\},$$

and

$$V_2 = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}}^2 : \det(\Psi(\lambda_1\lambda_2) - \lambda_2 I) = 0\},$$

and

$$V = V_1 \cap V_2.$$

Now since Φ and Ψ are matrix valued polynomial, the variety V is an algebraic variety in $\overline{\mathbb{D}}^2$. Note also that equation (5.1) implies that

$$\|p(T_1, T_2)\| \leq \sup_{(z_1,z_2)\in\partial V} |p(z_1, z_2)|,$$

and hence
$$\|p(T_1,T_2)\| \le \sup_{(z_1,z_2)\in V} |p(z_1,z_2)|,$$
for all $p \in \mathbb{C}[z_1,z_2]$. This completes the proof. $\square$

With the hypotheses of Theorem 5.1, let $(\mathcal{E}, U, P)$ be the triple corresponding to the pure isometric pair (M_Φ, M_Ψ). Let also assume that PU^* and $UP^\perp$ be completely non-unitary. It now follows from [8, Proposition 4.1] that $\Phi(z)$ and $\Psi(z)$, $z \in \mathbb{D}$, does not have any unimodular eigenvalue. Therefore
$$V \cap \{(\mathbb{D}\times\mathbb{T}) \cup (\mathbb{T}\times\mathbb{D})\} = \emptyset,$$
and hence
$$V \cap \partial\mathbb{D}^2 = V \cap \mathbb{T}^2.$$
This allows one to replace the algebraic variety V in Theorem 5.1 by an algebraic distinguished variety (see [1])
$$\tilde{V} = \tilde{V}_1 \cap \tilde{V}_2,$$
in $\mathbb{D}^2$, where
$$\tilde{V}_1 = \{(\lambda_1,\lambda_2) \in \mathbb{D}^2 : \det(\Phi(\lambda_1\lambda_2) - \lambda_1 I) = 0\},$$
and
$$\tilde{V}_2 = \{(\lambda_1,\lambda_2) \in \mathbb{D}^2 : \det(\Psi(\lambda_1\lambda_2) - \lambda_2 I) = 0\}.$$

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